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Orthogonal Grids Around Difficult Bodies

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A simple and easy-to-implement modification of the classical Theodorsen technique to map quasicircles onto circles is shown to allow contours of practical interest to be mapped directly onto circles, thus making the generation of orthogonal grids a routine matter in cases that, so far, appeared very hard to solve. Polygonal contours, elongated contours, wavy walls, and other oddities can be tranformed into circles or straight lines using the same technique in the form of a black-box subroutine. Many examples are presented.

Introduction

GRID generation is as popular as ever, and pros and cons of structured and unstructured grids are a subject of hot debates. The supporters of structured grids claim that computing on unstructured grids is cumbersome and requires extra computer resources. The supporters of unstructured grids point out that structured grids, most of the time obtained by numerical techniques, are ill fitted for accurate computations where the grid lines are far from being orthogonal and that concentration of grid lines, necessary in certain regions of the flowfield, has to be carried into other regions where much coarser meshes should be used. This paper intends to show that orthogonal grids can be obtained without a major effort in a much larger variety of problems as it has been believed so far.

The well-known properties at the background of conformal mapping show that the generation of an orthogonal grid around a closed contour depends on the mapping of the contour onto a circle via one analytic function or a sequence of analytic functions. Basically, conformal mapping techniques utilize the mappings defined by bilinear transformations, powers, logarithms and their inverse mappings, and the Joukowski and Kármán-Trefftz transformations. Proper combinations of such functions allow rather complicated contours of single bodies, multiple bodies, and cascades to be mapped on a single closed contour that is hoped to have a quasicircular shape. In many cases, the mapping is simple to conceive and to code; for example, an airfoil can be transformed into an almost circular contour via a single Kármán-Trefftz function, having a singular point at the trailing edge, another somewhere near the leading edge, and an exponent defined by the trailing-edge angle.

A final mapping is needed to transform a quasicircular shape into a circle. Theodorsen explained how to do it.[1] In what follows, we recall the essentials of Theodorsen's technique. Unfortunately, however, the procedure does not always work easily, and this has produced a certain general belief that orthogonal grids are hard to generate for practical applications.

Classic Theodorsen Mapping

Let z be the complex coordinate of a plane containing a quasicircle, and ζ the complex coordinate of a plane containing a circle. Theodorsen's mapping is the analytic function defined by

$$z = \zeta e^{f(\zeta)} \tag{1}$$

where

$$f(\zeta) = \sum_{n=0}^{\infty} (a_n + ib_n)\zeta^{-n} \tag{2}$$

It is convenient, but not restrictive, to define the circle as the unit circle, $|\zeta| = 1$. The origin is the center of the circle itself, and it is also the only singularity of $f(\zeta)$. Therefore, Eq. (1) can be used to map the entire z plane onto the entire ζ plane, except for an irrelevant region around the origin. (Theodorsen conceived his transformation to map the outside of a quasicircle onto the outside of a circle.) The quasicircle itself is also (unrestrictedly) located with its centroid at the origin, and the scale of the z plane is chosen to make the area contained within the quasicircle equal to π; consequently, the quasicircle is, in the mean, as close as possible to the circle. Finally, the real axes of the two planes are made to coincide.

Now, let r and ϕ be the modulus and argument of z, and θ the argument of the corresponding point ζ:

$$z = re^{i\phi}, \quad \zeta = e^{i\theta} \tag{3}$$

If the circle and quasicircle coincided, $f(\zeta)$ would vanish identically. In a practical case, $f(\zeta) \neq 0$ and the infinite sets of coefficients a_n and b_n have to be defined. Since we assumed that z and ζ are not too different from one another, the search is simplified. Indeed, from Eqs. (1) and (2),

$$\ln r = \Sigma(a_n \cos n\theta + b_n \sin n\theta) \tag{4a}$$

$$\phi = \theta + \Sigma(b_n \cos n\theta - a_n \sin n\theta) \tag{4b}$$

On state-of-the-art computers, the problem is solved by iteration, as follows:

A number N of points (equal to a power of 2) is chosen on the circle. As a first guess, it is assumed that $\theta = \phi$, and given values of $r(\phi)$ are interpolated at each θ. Then, Eq. (4a) is used to get the coefficients a_n and b_n for $n = 0,1,\ldots,N/2$ (a straightforward, fast-moving, Fourier transform subroutine does the job), and Eq. (4b) allows new values of ϕ to be evaluated. At the first attempt, obviously such new values turn out to be different from the guesses. An overall error is detected and the procedure is restarted, reinterpolating $\ln r$ at the new values of ϕ. The operation is completed when the error falls below a prescribed tolerance. When everything goes smoothly, a global error less than 10^{-8} can be reached in five or six steps with $N = 64$ or 128 without spending any appreciable time on a double-precision computer.

Need for a New Technique

In many cases, as we stated earlier, problems occur. Clearly, the assumption of closeness between the circle and the quasicircle is to be satisfied if we want the iteration to converge properly, but even for simple shapes the assumption seems to

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be hard to meet. Certain concepts, such as the term closeness, are not precisely defined; a good location for the second Kármán-Trefftz singularity, which, as stated earlier, should fall somewhere near the leading edge of an airfoil, at times may only result from a time-consuming empirical search. There are practical cases where, to get the Theodorsen technique to work properly, a certain number of attempts must be made, and this is definitively not good for a designer and may result in a considerable waste of time. Other features of the quasicircle may have an even worse effect. Indeed, a Fourier series as in Eqs. (4) can be used in practice only if it converges rapidly. For this to happen, the function to be expanded must be reasonably smooth to make the first harmonics relevant and the others negligible. Regions of strong gradients or strong curvature in $\ln r$ as a function of ϕ produce coefficients a_n and b_n that decrease very slowly with n, at times without changing their order of magnitude over the entire range $0 \leq n \leq N/2$. Consequently, the accuracy of the iterative calculation needed to compute any point in the ζ plane, given its counterpart in the z plane, can become so bad that the circle itself is distorted into a sequence of curlicues.

Promising Improvement

An intermediate shape, smoother than the given shape and closer to a circle, is necessary to improve the Fourier expansion. One way to get it consists of performing at least one additional mapping before applying Theodorsen's technique. Certain sharp corners, particularly if distributed symmetrically, are relatively easy to remove.[2] For other shapes (smooth but far from circles), the task is much more difficult; this is the problem that was mentioned earlier. Singularities (fractional powers) must be distributed at several points inside the closed contour; the choice of the number of singularities and of their strengths and locations is mostly dictated by personal experience and may succeed only by trial and error. It is definitively not a procedure easy to generalize, not a tool for the designer.

The present technique arises from the question, given a closed contour a, is it possible to interpolate a second contour b between a and the circle, using a simple, analytic function and then apply the Theodorsen technique to b? Obviously, b should be much closer to the circle than a. The shape of b should change gradually; at each step, the Theodorsen technique should provide the analytical bound between b and the circle, and an extrapolation would be performed to find the original contour a. Obviously, this will not be the case at the first attempt, generally, but we could try to iterate the procedure to get better results. In conclusion, we could set up a double iteration. During the entire calculation, neither the circle nor the given contour would change their shapes; the points on the circle would not be moved either (they would always be defined by the same values of θ), but the points on the given contour would be shifted from one step to another (as the points on the quasicircle do in the original Theodorsen technique). The shape of the intermediate contour, instead, would change ever so slightly, and this is why a second iteration à la Theodorsen is needed to find a new set of Fourier coefficients for the mapping of the auxiliary contour onto the circle.

The simplest analytical function relating three planes, z, ζ and z_1, is a linear one:

$$z_1 = \alpha\zeta + (1 - \alpha)z \tag{5}$$

where α is a constant, $0 < \alpha < 1$. For α close to 1, the auxiliary contour, defined by z_1, is very close to the circle; moreover, its shape will always be smooth, regardless of the presence of sharp corners in the original contour of the z plane. Therefore, the application of Theodorsen's technique should be easy. Another advantage of Eq. (5) is that we can invert it to get z from ζ and z_1 at the end of each Theodorsen cycle, without ambiguities, in a very direct way.

Defining

$$z_1 = \rho e^{i\psi} \tag{6}$$

we begin by choosing a number of points on the circle, equal to a power of 2 (this is necessary to implement the Theodorsen technique using a fast Fourier transform); we choose initial values of ϕ equal to the values of θ and interpolate initial locations of points on the given contour, defined by $Re^{i\theta}$. Then we apply Eq. (5) to find the locations of points z_1. The first inner iteration performs the Theodorsen mapping of z_1 onto ζ using the same procedure as described above (with the obvious substitutions of z_1, ρ, and ψ to z, r, and ϕ). Once the new values of z_1 have been found, new values for z are obtained by inverting Eq. (5):

$$z = (z_1 - \alpha\zeta)/(1 - \alpha) \tag{7}$$

The original contour in the z plane is reinterpolated to find new points with new values of r at the values of ϕ just found. The shape of the contour is obviously not changed, but the points are shifted on it. Generally, we will observe that the values of r, as computed from Eq. (7), do not coincide with the interpolated values. This means that our original choice of $\phi = \theta$ was not good. Therefore, Eq. (5) will be applied again using the interpolated values of r and the new ϕ to get a new shape for the auxiliary contour in the z_1 plane. The procedure will be repeated until convergence is reached.

Preliminary experiments with simple shapes show a definite improvement with respect to the original Theodorsen technique (see, e.g., Fig. 1). The original contour is a four-lobed figure. The three solid lines in Fig. 1 are a circle (a), the original contour (b), and the auxiliary contour as defined at the start of the procedure using $\alpha = 0.7$ (c). This figure has been drawn using only 32 points all around for clarity; naturally, all of the contours, and particularly the original one, appear as having corners that in reality do not exist. Note that, at the beginning of the first step, each set of corresponding points on the circle, the original contour, and the auxiliary contour are aligned with the center of the circle. The final situation is shown in Figs. 2 and 3. In Fig. 2, the three solid lines now

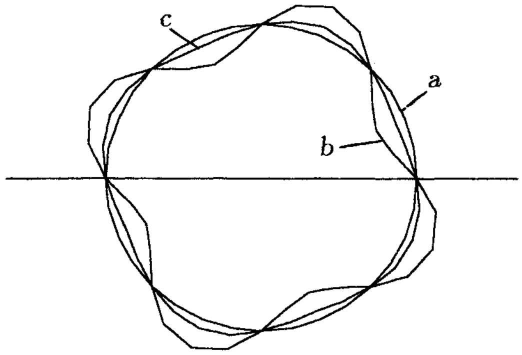


Fig. 1 Four-lobed contour, circle, and auxiliary contour.

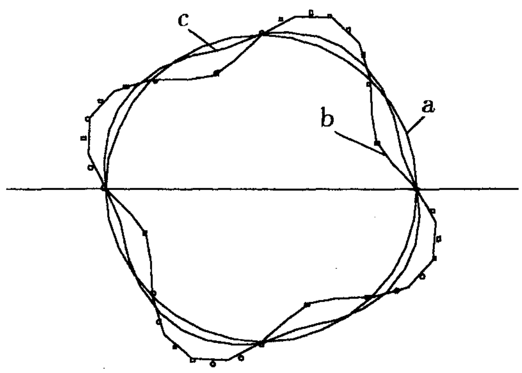


Fig. 2 Getting a four-lobed contour from a circle (32 points).

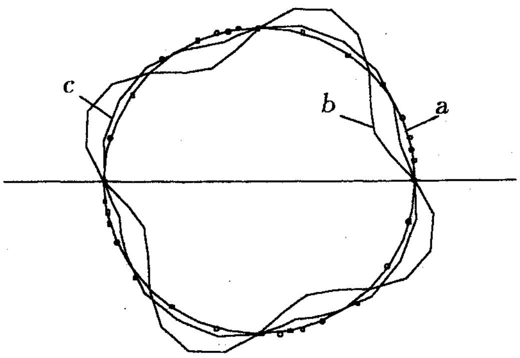


Fig. 3 Getting a circle from a four-lobed contour (32 points).

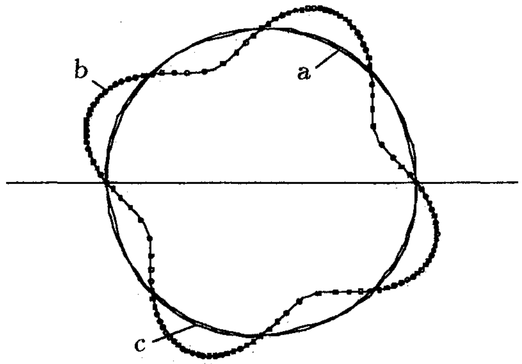


Fig. 4 Getting a four-lobed contour from a circle (128 points).

show, in addition to the circle, the auxiliary contour in its final shape and the original contour as it was in Fig. 1. If we consider 32 evenly spaced points on the circle and we use the present technique in reverse, we obtain the points denoted by small circles in Fig. 2. Note that they tend to accumulate where the contour, seen from the inside, is concave and to be more sparsely distributed where the contour is convex. The density of points increases with the curvature in the former case and decreases in the latter. Note also that the computed points seem not to fall on the original contour. This is not true; the original contour, as shown in Fig. 2, is defined by 32 points only, which are arbitrarily connected by straight segments; the computed points, instead, fall exactly on the contour defined analytically by $z = e^{i\phi}[1 - 2\sin(4\phi)]$. In Fig. 3, the operation is reversed; evenly spaced points are chosen on the original contour and small circles show their images on the ζ plane. If 128 points are used all around, the results, as shown in Fig. 4, are more appealing to the eye. For this calculation, α was chosen equal to 0.9, and the auxiliary contour appears very close to the circle. Several computations of this problem, with combinations of different values of α (between 0.7 and 0.95) and the number of points (32, 64, 128, and 256) showed no appreciable differences in numerical accuracy.

Mapping Contours with Corners

The present technique can also be used to map contours with corners, a more hostile environment. For example, consider a square centered at the origin and with a side equal to 3, trying to transform it into the unit circle with the exterior of the square mapped onto the exterior of the circle. First, let us consider some results (calculations were made again with combinations of values of α (= 0.6, 0.7, 0.8, and 0.9) and number of points (32, 64, and 128); except for the resolution of the plots, all runs are equivalent). Figure 5 was made with $\alpha = 0.9$ and 128 points. The inner solid line (a) is the unit circle, the contour surrounding it (b) is the auxiliary contour; the original square is shown by solid lines and the small circles have been obtained as images of evenly spaced points on the unit circle.

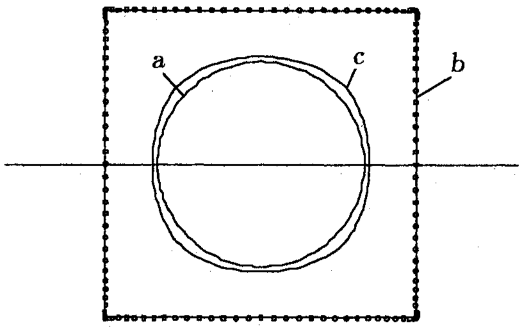


Fig. 5 Getting a square from a circle.

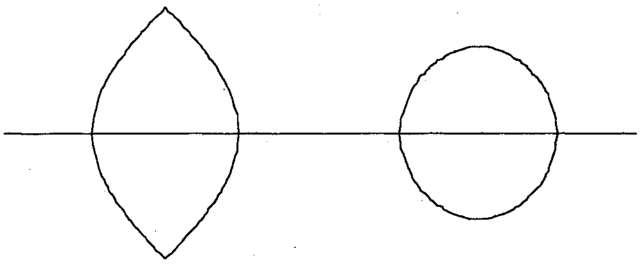


Fig. 6 Intermediate contours from a square, using Kármán-Trefftz mappings.

Again, we see a strong accumulation of computed points near the corners, where the curvature of the contour is the highest.

The remarkable, and unexpected, feature is that, with the exception of a slight inaccuracy in four points in the vicinity of each corner, the computed points fall on a square. This does not contradict the conservation of angles principle that is one of the fundamental properties of analytic functions and the backbone of conformal mapping. Indeed, the slight inaccuracies at the corners are what the analytic function resulting from the combination of Eqs. (1) and (5) has to do to turn around a 90-deg corner without violating the principle. The computed contour has to move out slightly in order to negotiate the corner with a smooth curve.

Let us consider the situation in a more formal way, recalling a property of analytic functions that helps to build practical conformal mappings. Any singularity of an analytic function can be relocated elsewhere by properly changing its type; for example, a first-order pole at $z = 1$ can be replaced by an essential singularity at the origin, defined by an infinite sum of poles:

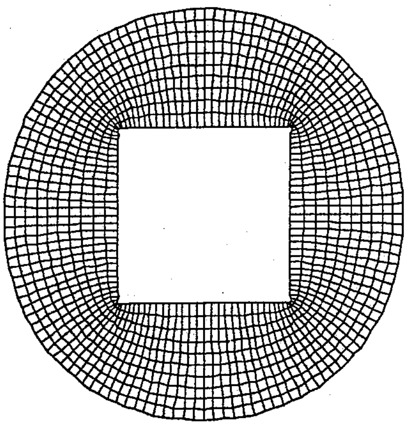
$$\frac{1}{z-1} = \sum_0^\infty \frac{1}{z^{n+1}} \tag{8}$$

a fractional power, $(z-1)^{-m}$, needed to smooth down a sharp corner at $z = 1$ ($m = 1/2$ for a Joukowski profile, $2/3$ for the square of Fig. 5) can be replaced by a similar formula:

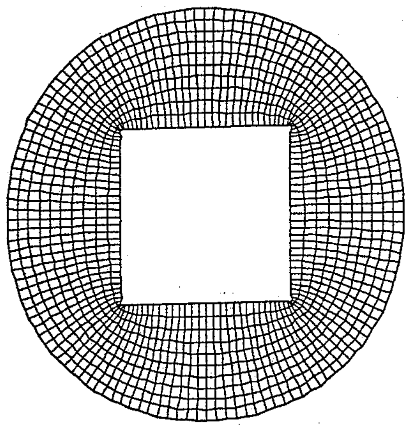
$$\frac{1}{(z-1)^m} = \frac{1}{z^m}\sum_0^\infty \binom{m}{n}(-z)^n \tag{9}$$

Therefore, there is no difference, in principle, between the two formulations; in practice, however, we truncate the series expansion, and such an arbitrariness shows up in a different behavior at the corners.

To illustrate the argument, we map the square of Fig. 5 onto the unit circle in three successive steps[2]: 1) a Kármán-Trefftz mapping with singularities at two opposite corners, 2) a second Kármán-Trefftz mapping based on the two remaining corners, and 3) a classic Theodorsen mapping. The two intermediate shapes are shown in Fig. 6. In Figs. 7, grids obtained by such a method and by the technique discussed in this paper are shown, and no difference can be noted.



a) Grid obtained by correct elimination of corners



b) Grid obtained by the present method

Fig. 7 Two orthogonal grids around a square.

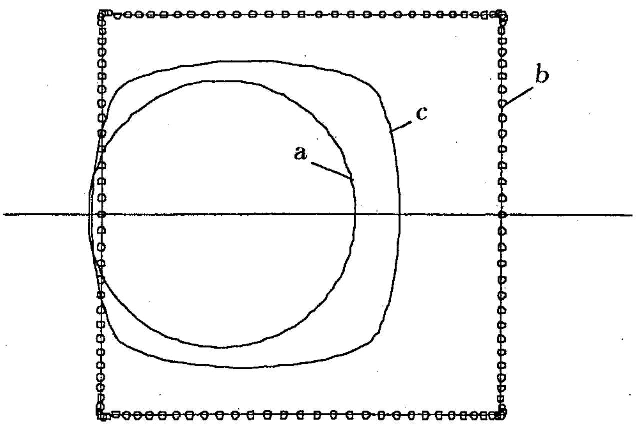


Fig. 8 Out-of-center square obtained from a circle.

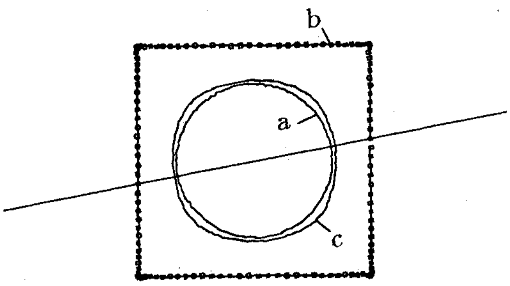


Fig. 9 Lopsided square obtained from a circle.

An off-center square can also be mapped onto a circle using the present technique; see Fig. 8 obtained using 128 points and $\alpha = 0.7$. A tilted square is mapped with similar ease (Fig. 9).

In conclusion, the present technique has the capability of mapping contours with any number of corners onto a circle with a single formulation because the singularities that are needed to smooth the corners down are automatically accumulated at the only singularity at the origin, provided by the Theodorsen transformation. In principle, the range of applications is limitless, encompassing odd-shaped polygons (Fig. 10) and even airfoils (Fig. 11); the latter is obtained starting from a symmetric Joukowski airfoil and making it into a nonsymmetric contour by multiplying the ordinates of the upper half by a constant factor. The last two cases have been obtained using the circle and the auxiliary quasicircle shown in the figures ($\alpha = 0.95$). An additional under-relaxation is used to get the new contour in the z_1 plane at each iteration; too strong a change in ρ can make the procedure locally unstable.

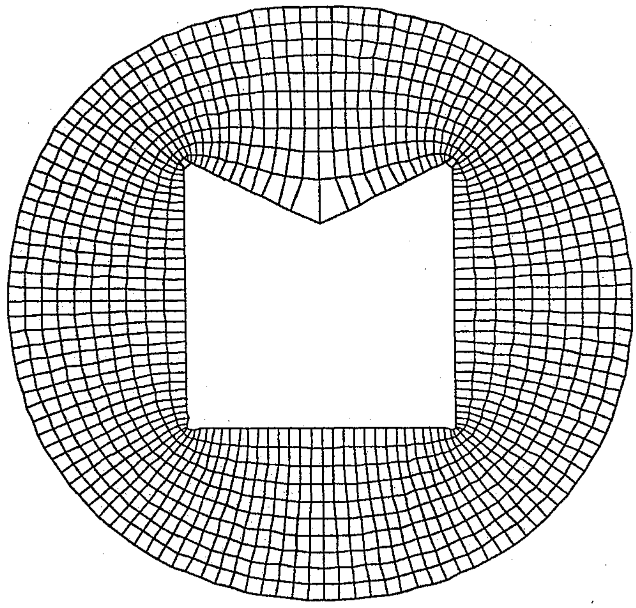


Fig. 10 Grid around a polygonal contour.

A portion β of the ρ computed according to Eq. (5) is combined with a portion $1 - \beta$ of the last value of ρ to make the change more gradual. Values of β ranging from 0.5 to 0.02 may be necessary. It is interesting to note that the overall process is not slowed down, even if β is very small. Indeed, an increase in the number of iteration steps is largely compensated by a dramatic decrease in the number of steps in each partial application of the Theodorsen mapping.

Despite the success of the technique, it should be kept in mind that the use of it as a substitute for gradual elimination of corners (as shown, e.g., in Figs. 6 and 7) is a brute-force approach to the imperfect solution of a problem that can be perfectly solved by the use of local singularities. It is important to know, however, that a powerful, and simple, technique is available as an auxiliary help, if not as a last resort. Minor irregularities in the immediate vicinity of a corner appear, in the present technique, as a consequence of truncations of the Fourier series and loss of accuracy in higher order terms; similar inaccuracies appear when corners are eliminated by analytical means. Regardless of the mapping procedure, indeed, the metric of the mapping is always singular at the corner; numerical inaccuracies inevitably appear when working too close to it. On the other hand, all flow problems must be given some local treatment at corners; the grid itself has no relevance at such points.

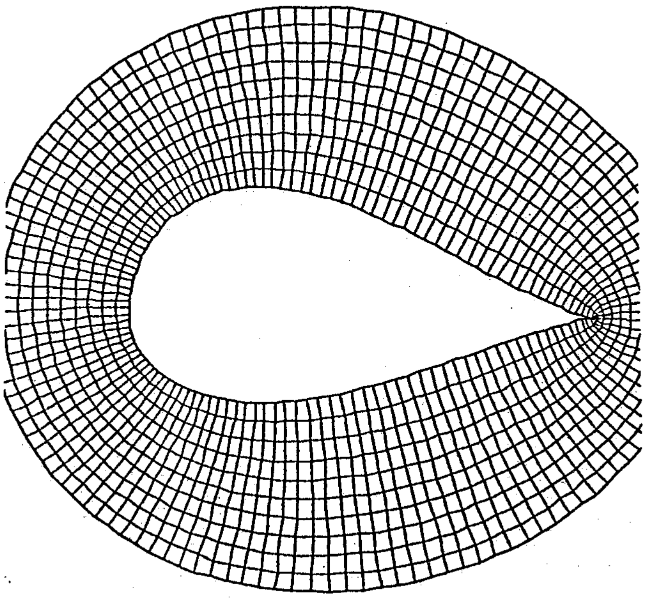


Fig. 11 Grid around a nonsymmetric, nonanalytic airfoil.

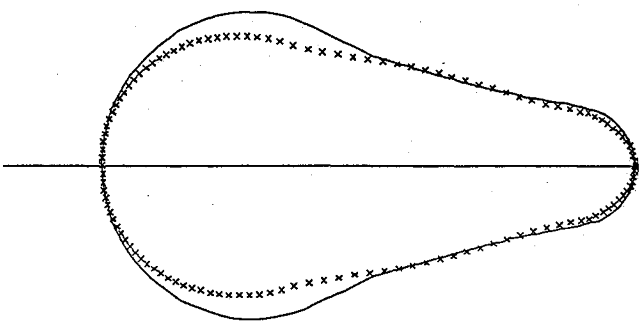


Fig. 12 First step in mapping a contour from a circle.

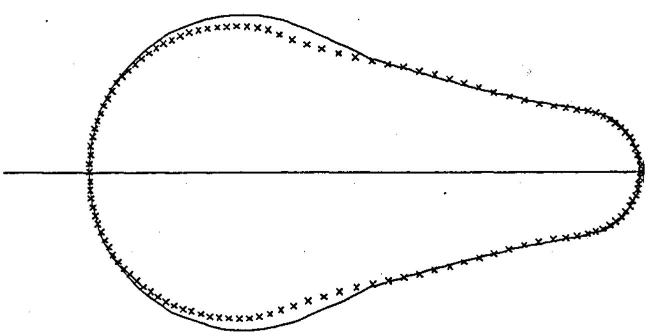


Fig. 13 Second step in mapping a contour from a circle.

More Applications

As we stated earlier, smooth but odd shapes may appear as a result of conformal mappings of odd-shaped bodies (such as slats) or cascades. The present technique seems to work very well on them. Here are two examples of increasing difficulty for shapes appearing in the conformal mapping of cascades of high stagger and high solidity. In both cases, $\alpha = 0.95$ and $N = 128$. In the first case, $\beta = 0.5$. For this case, we present three figures. In Fig. 12, the solid line is the given contour, and the crosses are points obtained from equidistant points on the circle at the first outer iteration. Figure 13 presents the situation after the second iteration, and Fig. 14 is the converged result after 10 iterations. In the second case, $\beta = 0.03$. In Fig. 15, we see the given contour, the circle, and the auxiliary contour (after convergence is reached) as solid lines; the

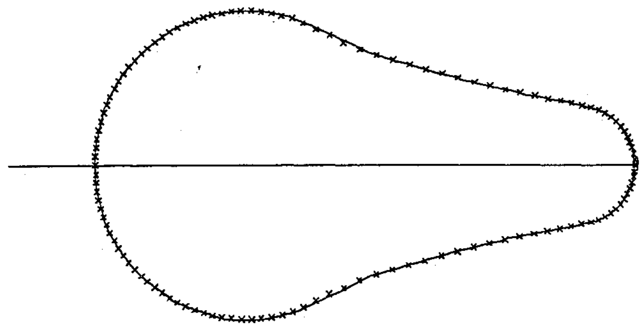


Fig. 14 Last step in mapping a contour from a circle.

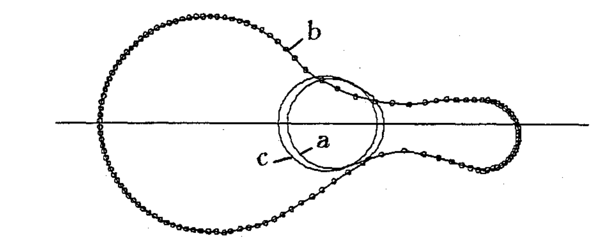


Fig. 15 Difficult contour mapped from a circle, the circle, and the auxiliary contour.

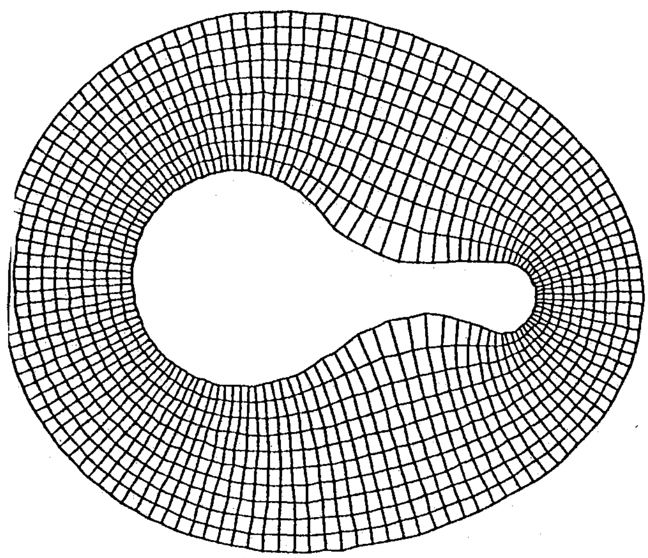


Fig. 16 Orthogonal grid around the contour of Fig. 15.

small circles are points obtained from an equidistribution of points on the circle, and we can see that they are well located on the original contour. Note how different the circle is from the given contour. Because of the oddity of the shape, many more outer iterations were needed. The computation was automatically stopped only when an overall error, less than 10^{-4} was reached, and this occurred after 203 iterations. It must be noted that practical convergence was reached far before that stage and that for about 150 iterations the Theodorsen evaluation was below the prescribed tolerance of 10^{-8} in a single step, so that the only operations to be performed were on the linear equation (7). Therefore, if the given body seems to have a difficult shape, one can play it safe by using a low value of β and letting the routine work for many steps because the time involved in the computation is not increased by a sizeable amount. Finally, Fig. 16 shows an orthogonal grid draped around the contour.

With such a flexibility and reliability as shown earlier, one can think of many possible extensions that could be hard to tackle using classic analytic tools. One of them is the semi-

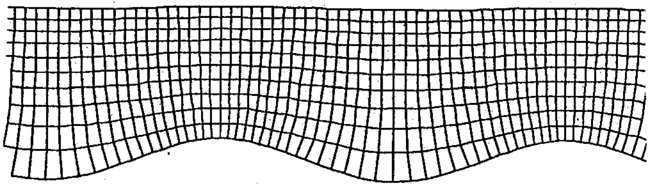


Fig. 17 Orthogonal grid above a wavy wall.

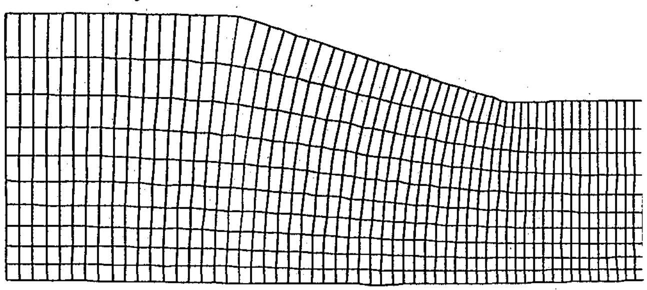


Fig. 18 Orthogonal grid in a duct.

infinite flowfield above a wavy wall. By using a preliminary transformation defined by the exponential function, one can map the wavy wall as a wavy shape superimposed to the unit circle, proceed to generate the grid around it as described in this paper, and then map it back onto the original plane (Fig. 17).

With more labor, but no conceptual changes, an orthogonal grid within a duct can be generated. Internal problems are generally harder to solve than external ones, and the mapping of a region such as the one shown in Fig. 18 onto a rectangle entails the application of Schwarz-Christoffel techniques. Whether the resulting integral is an elliptic integral of the second or third kind, or just an expression to be computed numerically, major difficulties appear. Here we can solve the problem by applying the present technique to map the lower wall onto a circle defined by $\zeta = Re^{i\theta}$ with $R > 1$, and the upper wall onto the unit circle. First we apply the exponential function to map the lower wall onto a circle of radius r_0 and the upper wall onto a quasicircle, close to the unit circle (and the region between them onto a quasiring). Then we proceed to apply the present technique in two successive steps. In the first step, the quasicircle is mapped onto the unit circle using Eqs. (2) and (5). This entails the existence of a singularity at the origin. In the second step, we need another singularity at infinity to map the other contour onto the unit circle (both singularities are obviously outside the ring in the ζ plane). Therefore, Eq. (2) must be replaced by

$$f(\zeta) = c_0 + id_0 + \sum_{n=1}^{\infty} (c_n + id_n)\zeta^n \tag{10}$$

and Eq. (5) can be retained in the same form. We can proceed to map the outer contour onto the unit circle using the same Fourier transform as used earlier to determine the coefficients c_0, d_0, c_n, and d_n [note that d_n is the same as b_n in Eqs. (4) but changed in sign; note also that in Eq. (4b) the sign in front of the summation must be changed]. At this stage, the image of the original straight line is the unit circle, and the image of the other contour is already sufficiently close to a circle of a smaller radius. A bilinear mapping may be used to move the inner quasicircle to be concentric with the unit circle. A final step can thus be taken, applying a conventional double Theodorsen routine to transform the two contours into really concentric circles, one of which is still the unit circle, whereas the radius of the other one has to be found by iteration.

Conclusions

We have seen that the simple addition of an auxiliary contour, linearly related to the original contour and a circle, allows the mapping of the former onto the latter and the consequent generation of an orthogonal grid to be achieved by applying the Theodorsen mapping to the auxiliary contour. A good choice for the interpolating parameter α is 0.95. In simple cases, the under-relaxing coefficient β can be left equal to 1. In general, one can let it be equal to 0.5, as a safety measure. In simple cases, 4–9 outer iterations are sufficient to reach a practical overall error of the order of 10^{-4}; each outer iteration may require 4–7 Theodorsen iterations, each with an overall error of the order of 10^{-8}. (To enforce a bigger tolerance or to use single precision is not recommended since it does not save computational time and minor errors in the inner calculations make the outer calculations erratic.) The technique seems to be a powerful and highly reliable tool for generating orthogonal grids about complicated bodies. We have presented here a few applications and examples that may prompt other types of extensions. The technique presented here is itself contained in a short program (including the auxiliary subroutines for splining, interpolating, and a fast Fourier transform); therefore, it can be called by any existing code and work as a black-box routine.

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